

Arithmetic of a singular K3 surface

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Abstract

This paper is concerned with the arithmetic of the elliptic K3 surface with configuration $[1,1,1,12,3^*]$. We determine the newforms and zeta-functions associated to X and its twists. We verify conjectures of Tate and Shioda for the reductions of X at 2 and 3.

Key words: elliptic surface, modularity, twisting, Tate Conjecture.

MSC(2000): 11F23, 11G25, 11G40, 14G10; 14J27, 14J28.

1 Introduction

This paper investigates the arithmetic of a particular singular K3 surface X over \mathbb{Q} , the extremal elliptic fibration with configuration $[1,1,1,12,3^*]$. First of all, we determine the corresponding weight 3 form (cf. [L, Ex. 1.6]) explicitly. For this, we calculate the action of Frobenius on the transcendental lattice by counting points and applying the Lefschetz fixed point formula. The proof is based on our previous classification of CM-forms with rational coefficients in [S2]. In fact, we only have to compute one trace.

Then we compute the zeta-function of the surface. This is used to study the reductions of X modulo some primes p . We emphasize that we are able to find a model with good reduction at 2. We subsequently verify conjectures of Tate and Shioda. The conjectures will be recalled in Section 4 and verified in Sections 5 - 7.

The final section is devoted to the twists of X . We show that these produce all newforms of weight 3 with rational coefficients and CM by $\mathbb{Q}(\sqrt{-3})$.

2 The extremal elliptic K3 fibration

There is a unique elliptic K3 surface X with configuration $[1,1,1,12,3^*]$ and with a section (No. 166 in [S-Z] and [S1, Tab. 2]). Since this arises as cubic base change of the extremal rational elliptic surface Y with singular fibres I_1^* , I_4 and I_1 , we shall start by studying this surface.

In [MP], an affine Weierstrass equation of this fibration was given as

$$Y' : y^2 = x^3 - 3(s-2)^2(s^2-3)x + s(s-2)^3(2s^2-9). \quad (1)$$

It has discriminant

$$\Delta = 16 \cdot 27(s-2)^7(s+2),$$

so the singular fibres are I_1^* above 2, I_1 above -2 and I_4 above ∞ .

We shall look for a model of Y over \mathbb{Q} which has everywhere good reduction. The fibre of Y' at ∞ has non-split multiplicative reduction, so $H_{\text{ét}}^2(Y', \mathbb{Q}_\ell)$ is ramified. Therefore we twist equation (1) over the splitting field $\mathbb{Q}(\sqrt{-3})$. Performing some elementary transformations (cf. [S3, IV.1]), we obtain the equation

$$Y : y^2 + sxy = x^3 + 2sx^2 + s^2x. \quad (2)$$

This elliptic surface has discriminant $\Delta = s^7(s + 16)$ and everywhere good reduction. In particular, all components of reducible fibres are defined over \mathbb{Q} . (For I_1^* , this can be derived from Tate's algorithm [Si, IV, §9].) Upon reducing mod 2, we obtain the equation from [I]. Y then only inherits the two singular fibres of types I_4 and I_1^* with wild ramification at the latter fibre.

We now come to the singular K3 surface X . Consider the cubic base change

$$\pi : s \mapsto s^3.$$

Via pull-back from Y , this gives rise to an extremal elliptic K3 surface. The resulting Weierstrass equation reads

$$y^2 + s^2 x y = x^3 + 2s x^2 + s^2 x. \quad (3)$$

This Weierstrass model has a D_7 resp. A_{11} singularity in the fibre above 0 resp. ∞ . By X , we denote the minimal desingularization. This elliptic fibration has configuration $[1,1,1,12,3^*]$. The reducible singular fibres I_3^* and I_{12} sit at 0 and ∞ . The three fibres of type I_1 can be found at the cube roots of -16 .

By construction, the absolute Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts trivially on the trivial lattice V of X , generated by the 0-section and the fibre components. Since X is extremal, tensoring V with \mathbb{Q} gives the corresponding statement for $NS(X)$.

The elliptic surface Y has everywhere good reduction. Furthermore, the base change π is nowhere degenerate upon reducing. Hence, the pull-back X can only have bad reduction at the prime divisors of the degree of π , i.e. at 3. Modulo 3, the Weierstrass model (3) obtains an additional A_2 singularity, so the reduction is in fact bad (cf. Sect. 5).

In terms of $H_{\text{ét}}^2(X, \mathbb{Q}_\ell)$, the ramification is reflected in the contribution of the transcendental lattice T_X , the orthogonal complement of $NS(X)$ in $H^2(X, \mathbb{Z})$. Here we consider it as a two-dimensional ℓ -adic Galois representation ρ . The reduction properties of X imply that ρ is only ramified at 3 and the respective prime ℓ . This agrees with the discriminant $d = d_{T_X}$ of X which is 3. To see this, recall that

$$d_{T_X} = -d_{NS(X)},$$

since $H^2(X, \mathbb{Z})$ is unimodular. Consider the trivial lattice V of X . Let U denote the hyperbolic plane, i.e. \mathbb{Z}^2 with intersection form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then

$$V = A_{11} \oplus D_7 \oplus U$$

has discriminant $d_V = -12 \cdot 4 = -48$. The Néron-Severi group $NS(X)$ is obtained from V by adding the sections. Here, X is extremal, so $MW(X)$ is finite. Hence

$$d_{NS(X)} = \frac{d_V}{|MW(X)|^2}.$$

Explicitly, we have $MW(X) \cong MW(Y)$, consisting of four elements (cf. Sect. 6). We obtain $d_{NS(X)} = -3$ and $d_{T_X} = 3$. Then the (reduced) intersection form on T_X can only be

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

3 The associated newform

By a result of Livné [L, Ex. 1.6], X (or T_X resp. ρ) has an associated newform of weight 3 with CM by $\mathbb{Q}(\sqrt{-3})$. Our aim is to determine this newform explicitly. Let p be a prime of good reduction, that is $p \neq 3$. Then

$$\det \rho(\text{Frob}_p) = \left(\frac{p}{3}\right) p^2. \quad (4)$$

To find the trace of ρ , we use the Lefschetz fixed point formula. We have already seen that $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ operates trivially on $NS(X)$. Hence, the Lefschetz fixed point formula gives

$$\#X(\mathbb{F}_p) = 1 + 20p + \text{tr } \rho(\text{Frob}_p) + p^2.$$

Using a computer program, we calculated the following traces at the first good primes.

p	2	5	7	11	13	17	19	23	29	31	37
$\text{tr } \rho$	0	0	-13	0	-1	0	11	0	0	-46	47

By inspection, these traces coincide with the Fourier coefficients of the newform $f = \sum_n a_n q^n$ of level 27 and weight 3 from [S2, Tab. 1]. We shall now prove that this holds at every prime:

Proposition 3.1

$$L(T_X, s) = L(f, s).$$

The proof makes use of Livné's modularity result for X . Let g denote the associated newform. Since g has CM by $\mathbb{Q}(\sqrt{-3})$, it is a twist of f by [S2, Thm. 3.4]. In our special situation, g is unramified outside 3 (i.e. it has level 3^r), since X has good reduction elsewhere. Hence we only have to compare the few possible newforms with such a level.

Let ψ_f denote the Größencharakter of conductor 3 which corresponds to f , and analogously ψ_g . Then [S3, Prop. II.11.1] and in particular the reasoning of section II.11.4 therein show that there are three possibilities in total:

$$\psi_g = \psi_f \quad \text{or} \quad \psi_g = \psi_f \otimes \left(\frac{3}{\cdot}\right)_3 \quad \text{or} \quad \psi_g = \psi_f \otimes \left(\frac{3}{\cdot}\right)_3^2. \quad (5)$$

Since the third residue symbol evaluates non-trivially at the factors of $p = 7$, it suffices to compare the Fourier coefficients (or traces) at 7. Thus $f = g$. A priori, this only guarantees that all but finitely many traces coincide. But here, the associated Galois representations ρ and ρ_f are simple, since their traces are not even. Hence Proposition 3.1 follows.

Corollary 3.2

The zeta-function of X is

$$\zeta(X, s) = \zeta(s) \zeta(s-1)^{20} L(f, s) \zeta(s-2).$$

Proof: We verify the corollary at every local Euler factor. On the one hand, at all primes $p \neq 3$, this follows from Proposition 3.1 and $\rho(X/\mathbb{Q}) = 20$.

On the other hand, the reduction X_3 at 3 is singular. Hence, the local Euler factor is only defined via the number of points of this singular variety over the fields \mathbb{F}_{3^r} , since the Lefschetz fixed point formula is not available.

The idea is that, with respect to the number of points, X_3 looks like \mathbb{P}^2 blown up in 19 rational points. Let us explain what we mean by this: We compare X_3 to Y_3 , the reduction of Y modulo 3. The base change π is purely inseparable mod 3. Hence, over any finite field \mathbb{F}_{3^r} , the smooth fibres of X_3 have the same number of points as the corresponding fibres of Y_3 . On the other hand, X_3 has ten additional \mathbb{P}^1 s in the singular fibres at 0 and ∞ . These are all defined over \mathbb{F}_3 . Thus

$$\#X_3(\mathbb{F}_{3^r}) = \#Y_3(\mathbb{F}_{3^r}) + 10 \cdot 3^r.$$

Recall that Y_3 is \mathbb{P}^2 , blown up in nine points. These points are all rational over \mathbb{F}_3 , since the absolute Galois group operates trivially. We deduce that

$$\#X_3(\mathbb{F}_{3^r}) = \#(\mathbb{P}^2(19))(\mathbb{F}_{3^r}).$$

This gives the local Euler factor

$$\zeta_3(X, T) = \frac{1}{(1-T)(1-3T)^{20}(1-3^2T)}.$$

The level 27 implies $L_3(f, s) = 1$ by classical theory. This can also be read off from [S2, Tab. 2] and the nebentypus $\chi_{-3} = \left(\frac{\cdot}{3}\right)$ of f . This completes the proof of Corollary 3.2.

4 The conjectures for the reductions

In this section, we shall discuss conjectures of Tate, Shioda and Artin for smooth projective surfaces over finite fields. In particular, these conjectures apply to (supersingular) reductions of varieties defined over number fields.

Corollary 3.2 will be very useful: If X has good reduction at p , then the corollary gives the local ζ -function of the smooth variety X/\mathbb{F}_p . Explicitly, let $p \neq 3$. We obtain

$$P_2(X/\mathbb{F}_p, T) = \det(1 - \text{Frob}_p T; H_{\text{ét}}^2(X/\overline{\mathbb{F}}_p, \mathbb{Q}_\ell)) = (1 - pT)^{20} (1 - a_p T + \chi_{-3}(p) p^2 T^2).$$

This is exactly where the Tate conjecture enters. To formulate it, consider a finite field k and a smooth projective variety Z/k . Define the Picard number of Z over k

$$\rho(Z/k) = \text{rk } NS(Z)^{\text{Gal}(\overline{k}/k)}.$$

We employ the convention $\rho(Z) = \rho(Z/\overline{k})$ when the field of definition of Z is understood.

Conjecture 4.1 (Tate [T1, (C)], [T2])

Let $q = p^r$ and Z/\mathbb{F}_q a smooth projective variety. Denote the order of the zero of $P_2(Z/\mathbb{F}_q, T)$ at $T = \frac{1}{q}$ by u . Then $u = \rho(Z/\mathbb{F}_q)$.

The Tate conjecture is known for elliptic K3 surfaces with a section in characteristic $p > 3$ [T2, Thm. (5.6)].

We can consider the Weierstrass model (3) over any $\mathbb{F}_q, q = p^r$. Denote the minimal resolution of the A_n and D_m singularities by X/\mathbb{F}_q . This surface coincides with the reduction X_p of X if and only if p is a good prime (i.e. $p \neq 3$). On the other hand, X_3 contains an A_2 singularity, so it is not smooth. The desingularization X/\mathbb{F}_3 will be sketched in the next section.

Proposition 4.2

Let p be a prime. Consider X/\mathbb{F}_p . Then

$$\rho(X/\mathbb{F}_p) = \begin{cases} 20, & \text{if } p \equiv 1 \pmod{3}, \\ 21, & \text{if } p \equiv 0, 2 \pmod{3}. \end{cases}$$

If $p > 3$, Proposition 4.2 follows from the (known) Tate conjecture. The proofs for $p = 2$ and 3 will be given in the following three sections. We will also verify a conjecture of Shioda which is introduced in the following.

Let L be a number field and Z a singular K3 surface over L . If \mathfrak{p} is a prime of L , denote the residue field of L at \mathfrak{p} by $L_{\mathfrak{p}}$. We call \mathfrak{p} supersingular if and only if $Z/L_{\mathfrak{p}}$ is supersingular (i.e. $\rho(Z/\overline{L}_{\mathfrak{p}}) = 22$).

Shioda's conjecture concerns the surface $Z/L_{\mathfrak{p}}$ at a supersingular prime \mathfrak{p} . We can compare two lattices of rank two: On the one hand, we have the transcendental lattice T_Z of Z/L . On the other hand we can use the natural embedding

$$NS(Z/\overline{L}) \subseteq NS(Z/\overline{L}_{\mathfrak{p}})$$

to define the orthogonal complement

$$T_{\mathfrak{p}} = NS(Z/\overline{L})^{\perp} \subset NS(Z/\overline{L}_{\mathfrak{p}}).$$

Conjecture 4.3 (Shioda [Sh2, Conj. 4.1])

Let Z be a singular K3 surface over a number field L and \mathfrak{p} a supersingular prime. Then, the two lattices T_Z and $T_{\mathfrak{p}}$ are similar.

In other words, the claim is that $T_{\mathfrak{p}}$ is isomorphic to $T_Z(-m)$ for some $m \in \mathbb{Q}_{>0}$. We will verify this conjecture for the extremal elliptic K3 fibration X at the primes 2 and 3 in the next three sections. This will be achieved by finding explicit generators of the respective $T_{\mathfrak{p}}$. As a by-product, we will thus verify the following

Theorem 4.4 (Artin [A, (6.8)])

Let Z be a supersingular K3 surface over a finite field k of p^r elements. If r is odd, then $\text{Gal}(\bar{k}/k)$ operates non-trivially on $NS(Z)$.

At the time of this paper, the Brauer group $Br(Z)$, if finite, was only known to have cardinality a square or twice a square. Hence, Artin was only able to prove the above result for odd p . Recently, the second alternative has been ruled out in [LLR]. Hence, Artin's argumentation in [A, §6] also applies to $p = 2$.

Note that Theorem 4.4 agrees perfectly with Proposition 4.2 for our surface X . In fact, the known part of Tate's Conjecture 4.1 shows that at a supersingular prime $p > 3$ (i.e. $p \equiv -1 \pmod{3}$),

$$\rho(X/\mathbb{F}_{p^2}) = 22.$$

5 X/\mathbb{F}_3

In this section, we shall consider X/\mathbb{F}_3 . This will be special because π is purely inseparable modulo 3. As a consequence, the base change X/\mathbb{F}_3 from Y/\mathbb{F}_3 via π has only three singular fibres. They have types I_3^* , I_{12} and I_3 . In particular, the elliptic fibration X/\mathbb{F}_3 is extremal (cf. the classification of [I]) and supersingular. By construction, it also is unirational. We emphasize that the reduction X_3 is not smooth.

We shall now verify Tate's and Shioda's conjectures for X/\mathbb{F}_3 . Consider the 4-torsion sections of X/\mathbb{F}_3 which come from X/\mathbb{Q} upon reducing (see Section 6 for a detailed study). These sections meet the O -component of the I_3 fibre. Denote the other components of the I_3 fibre (not meeting O) by Θ_1, Θ_2 . By construction, they also do not meet the generators of the trivial lattice V of X/\mathbb{Q} (embedded into $NS(X/\mathbb{F}_3)$). Hence, they are orthogonal to $NS(X/\mathbb{Q}) \subset NS(X/\mathbb{F}_3)$. We claim that this is already all of $NS(X/\mathbb{F}_3)$:

Lemma 5.1

Let A_2 denote the root lattice generated by Θ_1 and Θ_2 . Then

$$NS(X/\mathbb{F}_3) \cong NS(X/\mathbb{Q}) \oplus A_2.$$

The proof of this lemma will be directly derived from the following classical

Theorem 5.2 (Artin, Rudakov-Šafarevič)

Let X be a supersingular K3 surface over a field of characteristic p . Then

$$\text{discr } NS(X) = -p^{2\sigma_0} \quad \text{for some } \sigma_0 \in \{1, \dots, 10\}.$$

Here, σ_0 is called the Artin invariant.

As a consequence, in our situation, we have $\text{discr } NS(X/\mathbb{F}_3) = -3^{2\sigma_0}$ for some $\sigma_0 \in \{1, \dots, 10\}$. But then, the above sublattice $NS(X/\mathbb{Q}) \oplus A_2$ of $NS(X/\mathbb{F}_3)$ has rank 22 and discriminant -9 . Since this is the maximal possible discriminant, we deduce the equality of the two lattices. This proves Lemma 5.1.

Corollary 5.3

The supersingular K3 surface X/\mathbb{F}_3 has Artin invariant $\sigma_0 = 1$.

We shall now prove Proposition 4.2 for $p = 3$. Since $\rho(X/\mathbb{Q}) = 20$, the reduction of $NS(X/\mathbb{Q})$ is clearly generated by divisors over \mathbb{F}_3 . Using Lemma 5.1, we only have to study the field of definition of the two further generators Θ_1, Θ_2 of $NS(X/\mathbb{F}_3)$.

For this, consider the elliptic curve over $\mathbb{F}_3(s)$ given by equation (3). It has non-split multiplicative reduction at $s = -1$. More precisely, the components Θ_1 and Θ_2 are conjugate in $\mathbb{F}_3(\sqrt{-1})$. In particular, their sum is defined over \mathbb{F}_3 , while the difference has eigenvalue

-1 with respect to the conjugation. Hence, we deduce the claim of Proposition 4.2 that $\rho(X/\mathbb{F}_3) = 21$. This agrees with the Tate conjecture and Theorem 4.4. We also see that $\rho(X/\mathbb{F}_9) = 22$.

Finally, we come to Shioda's Conjecture. By Lemma 5.1, we have

$$T_3 = NS(X/\mathbb{Q})^\perp = A_2 \subset NS(X/\overline{\mathbb{F}}_3).$$

Since A_2 has intersection matrix $\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$, we deduce the validity of Conjecture 4.3 with $m = 1$.

6 $NS(X)$

To verify the conjectures for the reduction X_2 , we need a better knowledge of $NS(X) = NS(X/\mathbb{Q})$. To be precise, we want to express the sections of the original fibration X over \mathbb{Q} in terms of $V \otimes_{\mathbb{Z}} \mathbb{Q}$. This is possible since the fibration is extremal, such that

$$MW(X) \cong NS(X)/V$$

is only torsion. In terms of equation (3), the sections of the elliptic fibration X are given by:

$$MW(X) \cong \mathbb{Z}/4 = \langle P \rangle = \langle (-s, 0) \rangle = \{(-s, 0), (0, 0), (-s, s^3), O\}.$$

They can be derived from the sections of Y' , as given in [MP], by following them through the base and variable changes. We will later see that $MW(X)$ gives all torsion-sections of $X/\overline{\mathbb{F}}_2$ upon reducing.

We want to express the section P as a \mathbb{Q} -divisor in $V \otimes_{\mathbb{Z}} \mathbb{Q}$. This can be achieved by determining the precise components of the singular fibres which P intersects.

In the I_{12} fibre above ∞ , we number the components cyclically $\Theta_0, \dots, \Theta_{11}$ such that Θ_0 meets O . The fibre of type I_3^* above 0 is sketched in Figure 1. As usual, C_0 denotes the component meeting O . The components D_i have multiplicity two. The freedom of renumbering components is killed by the following elementary observation:

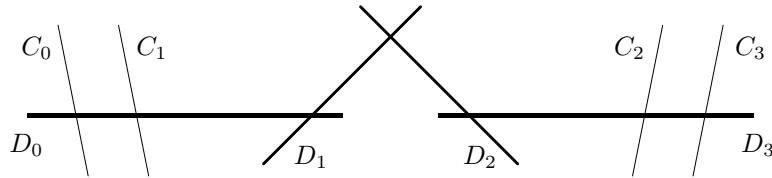


Fig. 1: The fibre of type I_3^* at 0

Lemma 6.1

Up to renumbering, the section P meets the components C_3 and Θ_9 .

Lemma 6.1 enables us to describe P in terms of the generators of the trivial lattice V . Since P is given by polynomials (of low degree), it does not meet O . We will use the \mathbb{Q} -divisors

$$\begin{aligned} A &= \frac{1}{4}(\Theta_1 + 2\Theta_2 + \dots + 9\Theta_9 + 6\Theta_{10} + 3\Theta_{11}) \quad \text{and} \\ B &= \frac{1}{4}(2C_1 + 4D_0 + 6D_1 + 8D_2 + 10D_3 + 5C_2 + 7C_3). \end{aligned}$$

Corollary 6.2

In $V \otimes_{\mathbb{Z}} \mathbb{Q}$, the section P is given as

$$P = O + 2F - A - B.$$

Proof: Since $4P \equiv 0 \pmod{V}$, it suffices to check the following intersection numbers:

$$(P.O) = 0, (P.F) = 1, (P.\Theta_i) = \delta_{i,9}, (P.D_j) = 0, (P.C_j) = \delta_{j,3} \text{ and } (P^2) = -2.$$

7 X/\mathbb{F}_2

In this section, we shall consider the elliptic K3 surface X/\mathbb{F}_2 . This coincides with the reduction X_2 of X at 2. Reducing equation (3) modulo 2, we obtain the affine equation

$$X/\mathbb{F}_2: y^2 + s^2 xy = x^3 + s^2 x. \quad (6)$$

Recall that this has only two singular fibres. They sit at 0 and ∞ and have types I_3^* and I_{12} . Since 2 stays inert in $\mathbb{Q}(\sqrt{-3})$, the local L -factor is given by

$$P_2(X/\mathbb{F}_2, T) = (1 - 2T)^{21} (1 + 2T)$$

because of Corollary 3.2. In accordance with the Tate conjecture, we shall prove that X is a supersingular K3 surface. To be precise, we claim

Proposition 7.1

X/\mathbb{F}_2 is a supersingular K3 surface with Picard numbers

$$\rho(X/\mathbb{F}_2) = 21 \text{ and } \rho(X/\mathbb{F}_4) = 22.$$

This proposition completes the proof of Proposition 4.2. It will be established by finding explicit generators for $NS(X/\overline{\mathbb{F}}_2)$. Additionally to the reduction of $NS(X/\mathbb{Q})$, we need two generators. By the formula of Shioda-Tate, these can be given as sections of the elliptic fibration X/\mathbb{F}_2 .

In detail, we compute some additional sections of $X/\overline{\mathbb{F}}_2$ which are not derived from $MW(X/\mathbb{Q})$ by way of reduction. Then we determine two among them which supplement $NS(X/\mathbb{Q})$ to generate $NS(X/\overline{\mathbb{F}}_2)$. Let α be a generator of \mathbb{F}_4 over \mathbb{F}_2 , i.e. $\alpha^2 + \alpha + 1 = 0$. Among others, we found the following sections. Here, the inverse refers to the group law on the generic fibre.

	section	inverse
	$Q = (1, 1)$	$(1, 1 + s^2)$
	(s^2, s^2)	$(s^2, s^2 + s^4)$
	$(s + s^3, s^3 + s^4)$	$(s + s^3, s^4 + s^5)$
$R =$	$(s + \alpha s^3, \alpha^2 s^4 + \alpha s^5)$	$(s + \alpha s^3, s^3 + \alpha^2 s^4)$
	$(1 + s^4, 1 + \alpha s^2 + \alpha^2 s^6)$	$(1 + s^4, 1 + \alpha^2 s^2 + \alpha s^6)$

We shall now prove that $MW(X/\overline{\mathbb{F}}_2)$ has rank 2 and can be generated by the sections Q and R together with the torsion-section P . This will be achieved with the help of the *height pairing* on the Mordell-Weil group, as introduced by Shioda in [Sh1]. Let V denote the trivial lattice of X/\mathbb{F}_2 . This is exactly the reduction of the trivial lattice of X/\mathbb{Q} . The height pairing is defined via the orthogonal projection

$$\varphi: MW(X/\overline{\mathbb{F}}_2) \rightarrow V^\perp \otimes_{\mathbb{Z}} \mathbb{Q} \subset NS(X/\overline{\mathbb{F}}_2) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

This projection only uses the information which (simple) component of a reducible fibre a section meets. Let (\cdot, \cdot) denote the intersection form on $NS(X/\overline{\mathbb{F}}_2) \otimes_{\mathbb{Z}} \mathbb{Q}$. Shioda's height pairing is defined by

$$\begin{aligned} \langle \cdot, \cdot \rangle: MW(X/\overline{\mathbb{F}}_2) \times MW(X/\overline{\mathbb{F}}_2) &\rightarrow \mathbb{Q} \\ \mathcal{P} \quad \quad \quad \mathcal{Q} &\mapsto -(\varphi(\mathcal{P}), \varphi(\mathcal{Q})). \end{aligned}$$

The height pairing is symmetric and bilinear. It induces the structure of a positive definite lattice on $MW(X/\overline{\mathbb{F}}_2)/MW(X/\overline{\mathbb{F}}_2)_{\text{tor}}$ (the Mordell-Weil lattice). For $\mathcal{P}, \mathcal{Q} \in MW(X/\overline{\mathbb{F}}_2)$, Shioda shows that

$$\langle \mathcal{P}, \mathcal{Q} \rangle = \chi(\mathcal{O}_X) - (\mathcal{P}, \mathcal{Q}) + (\mathcal{P}, O) + (\mathcal{Q}, O) - \sum_v \text{contr}_v(\mathcal{P}, \mathcal{Q}).$$

Here the sum runs over the cusps and contr_v can be computed strictly in terms of the components of the singular fibres which \mathcal{P} and \mathcal{Q} meet [Sh1, Thm. 8.6] (cf. also [Sh1, (8.16)]). The following lemma is easily checked:

Lemma 7.2

Q meets C_0 and Θ_8 while R intersects C_3 and Θ_1 .

We are now in the position to compute the projections $\varphi(Q)$ and $\varphi(R)$. However, we will postpone their explicit calculation, since we will only need this for the explicit verification of Shioda's Conjecture 4.3. Here, we can use Shioda's results from [Sh1] to prove that $NS(X/\mathbb{F}_3)$ is generated by the trivial lattice V and the sections P, Q and R .

Lemma 7.3

Let Q, R be the sections of X/\mathbb{F}_4 as specified above. Then

$$\begin{pmatrix} \langle Q, Q \rangle & \langle Q, R \rangle \\ \langle Q, R \rangle & \langle R, R \rangle \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}.$$

Proof: Since they are given by polynomials (of low degree), Q and R do not meet O . For the self intersection numbers, we thus only miss the contributions from the singular fibres. These are derived from [Sh1, (8.16)] with the help of Lemma 7.2:

$$\begin{aligned} \langle Q, Q \rangle &= 4 - \frac{4 \cdot 8}{12} = \frac{4}{3} \\ \langle R, R \rangle &= 4 - \frac{7}{4} - \frac{11}{12} = \frac{4}{3}. \end{aligned}$$

For the remaining entry, we furthermore have to analyze the intersection of Q and R . We have to find the common zeroes of

$$\begin{aligned} \alpha s^3 + s + 1 &= \alpha(s + \alpha^2)(s^2 + \alpha^2 s + 1) \quad \text{and} \\ \alpha s^5 + \alpha^2 s^4 + 1 &= \alpha(s + 1)(s + \alpha^2)(s^3 + \alpha^2 s + 1). \end{aligned}$$

The factors in the above factorization are irreducible over \mathbb{F}_4 . Hence, the only common zero is $s = \alpha^2$. Since this occurs with multiplicity one, the intersection is transversal. Hence, we deduce

$$\langle Q, R \rangle = 2 - 1 - \frac{1 \cdot 4}{12} = \frac{2}{3}.$$

This finishes the proof of Lemma 7.3.

Proposition 7.4

$NS(X/\mathbb{F}_2)$ has discriminant -4 . It is generated by the trivial lattice V and the sections P, Q and R , so $MW(X/\mathbb{F}_2) = MW(X/\mathbb{F}_4) = \langle P, Q, R \rangle$.

Proof: Consider the lattice N which is generated by the trivial lattice V and the sections P, Q and R . Clearly, this is a sublattice of $NS(X/\mathbb{F}_2)$. We can identify $N' = \langle V, P \rangle \cong NS(X/\mathbb{Q}) \subset N$. Recall that this sublattice has discriminant -3 .

Then we use the orthogonal projection φ from $NS(X/\mathbb{F}_2)$ to $V^\vee \otimes_{\mathbb{Z}} \mathbb{Q}$. Since $P \in V \otimes_{\mathbb{Z}} \mathbb{Q}$, it follows that

$$\text{discr } N = (\text{discr } N') \det \begin{pmatrix} (\varphi(Q), \varphi(Q)) & (\varphi(Q), \varphi(R)) \\ (\varphi(Q), \varphi(R)) & (\varphi(R), \varphi(R)) \end{pmatrix}.$$

By Lemma 7.3, the matrix has determinant $\frac{4}{3}$. Hence, N has discriminant -4 and in particular rank 22. Since both values are maximal possible (cf. Thm. 5.2), we obtain $N = NS(X/\mathbb{F}_2)$. The claim $MW(X/\mathbb{F}_2) = \langle P, Q, R \rangle$ then follows from the formula of Shioda-Tate. This proves Proposition 7.4.

Proposition 7.4 implies Proposition 7.1. It verifies the Tate conjecture for X/\mathbb{F}_{2^r} with any $r \in \mathbb{N}$. We also deduce the validity of Theorem 4.4. Furthermore, we have seen that X_2 has Artin invariant $\sigma_0 = 1$.

We conclude this section by verifying Conjecture 4.3 for X/\mathbb{Q} and its reduction at 2. Note that it suffices to consider $NS(X/\mathbb{Q})$ and $NS(X/\mathbb{F}_4)$. The dual of $NS(X/\mathbb{Q})$ will always refer to the embedding into $NS(X/\mathbb{F}_4)$.

To give the explicit form of $\varphi(Q)$ and $\varphi(R)$, recall the \mathbb{Q} -divisors A and B from Section 6. The projections are easily computed as

$$\begin{aligned}\varphi(Q) &= Q - O - 2F + \frac{1}{3}(\Theta_1 + 2\Theta_2 + \dots + 8\Theta_8 + 6\Theta_9 + 4\Theta_{10} + 2\Theta_{11}) \\ \varphi(R) &= R - O - 2F + B + \frac{1}{12}(11\Theta_1 + 10\Theta_2 + \dots + \Theta_{11}).\end{aligned}$$

Since the denominators are divisible by 3, $\varphi(Q), \varphi(R) \notin NS(X/\mathbb{F}_4)$. In other words,

$$\langle \varphi(Q), \varphi(R) \rangle \supsetneq NS(X/\mathbb{Q})^\perp \supsetneq \langle 3\varphi(Q), 3\varphi(R) \rangle. \quad (7)$$

We claim that

$$NS(X/\mathbb{Q})^\perp = \langle 3\varphi(Q), \varphi(Q) + \varphi(R) \rangle = \langle 3\varphi(R), \varphi(Q) + \varphi(R) \rangle. \quad (8)$$

Since the ranks are two, the inequality of (7) implies that the claim is equivalent to

$$\varphi(Q) + \varphi(R) \in NS(X/\mathbb{F}_4).$$

Since $P \equiv -A - B \pmod{V}$, it suffices to show that

$$\varphi(Q) + \varphi(R) \equiv A + B \pmod{\langle V, Q, R \rangle}.$$

Explicitly, we have

$$\begin{aligned}\varphi(Q) + \varphi(R) &= Q + R - 2O - 4F + \frac{1}{12}(11\Theta_1 + 10\Theta_2 + \dots + \Theta_{11}) \\ &\quad + B + \frac{1}{3}(\Theta_1 + 2\Theta_2 + \dots + 8\Theta_8 + 6\Theta_9 + 4\Theta_{10} + 2\Theta_{11}) \\ &\equiv B + \frac{1}{12} \sum_j (12 - j)\Theta_j + \frac{1}{3} \sum_j j\Theta_j \\ &= B + \frac{1}{12} \sum_j (12 - j + 4j)\Theta_j \\ &\equiv B + \frac{1}{4} \sum_j j\Theta_j \equiv A + B \pmod{\langle V, Q, R \rangle}.\end{aligned}$$

This proves claim (8). We shall now verify Conjecture 4.3 for X/\mathbb{Q} and its reduction at 2. Here $T_2 = \langle 3\varphi(Q), \varphi(Q) + \varphi(R) \rangle$ with intersection form

$$\begin{aligned}&\begin{pmatrix} (3\varphi(Q).3\varphi(Q)) & (3\varphi(Q).\varphi(Q) + \varphi(R)) \\ (3\varphi(Q).\varphi(Q) + \varphi(R)) & (\varphi(Q) + \varphi(R).\varphi(Q) + \varphi(R)) \end{pmatrix} \\ &= - \begin{pmatrix} 12 & 6 \\ 6 & 4 \end{pmatrix} = -2 \begin{pmatrix} 6 & 3 \\ 3 & 2 \end{pmatrix} \sim -2 \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.\end{aligned}$$

Hence, Conjecture 4.3 holds with $m = 2$.

Remark 7.5

In [DK], Dolgachev and Kondō give several models of the supersingular K3 surface in characteristic 2 with Artin invariant $\sigma_0 = 1$. The most natural might be the quasi-elliptic fibration

$$Z : y^2 = x^3 + t^2x + t^{11}.$$

The only exceptional fibre of this fibration has type I_{16}^* . This shows that

$$NS(Z) = U \oplus D_{20}.$$

We realize a \mathbb{F}_2 -isomorphic fibration with the above exceptional fibre in terms of our model X/\mathbb{F}_2 by giving an effective divisor L of type I_{16}^* . Denote by ιR the conjugate of the section R in \mathbb{F}_4 and by $[3P]$ the section $(-s, s^3)$ (the inverse of P with respect to the group law on the fibre). Recall that both, R and ιR meet C_3 and Θ_1 . $[3P]$ meets C_2 and Θ_3 (cf. Fig. 2). Thus the required divisor L (over \mathbb{F}_2) is given by

$$\begin{aligned}L &= R + \iota R + 2(\Theta_1 + \Theta_0 + \Theta_{11} + \Theta_{10} + \Theta_9 + \Theta_8 + \Theta_7 + \Theta_6 + \Theta_5 \\ &\quad + \Theta_4 + \Theta_3 + [3P] + C_2 + D_3 + D_2 + D_1 + D_0) + C_0 + C_1.\end{aligned}$$

8 Twisting

This section concludes the investigation of the extremal elliptic K3 surface X by commenting on twisting. Our aim is to show that the twists of the associated newform f with rational coefficients are in correspondence with twists of X . This is non-trivial, since f admits cubic twists.

For the quadratic twisting, this is well-known. For instance, it follows from point counting that the twist of X over $\mathbb{Q}(\sqrt{d})$ corresponds to the quadratic twist $f \otimes \left(\frac{d}{\cdot}\right)$. Note that there is a model of this twist over \mathbb{Q} with bad reduction exactly at 3 and the primes dividing the discriminant of $\mathbb{Q}(\sqrt{d})$.

We shall now come to the cubic twists. For X , they can be achieved by considering the base change

$$\pi_d : s \mapsto d s^3$$

instead of the original π . Here, we can restrict to positive cube-free d . Let $X^{(d)}$ denote the pull-back from the rational elliptic surface Y by π_d . An affine model is given by

$$X^{(d)} : y^2 + d^2 s^2 x y = x^3 + 2s x^2 + s^2 x. \quad (9)$$

Of course, $X^{(d)}$ has the same configuration of singular fibres as X and also inherits the trivial action of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $NS(X^{(d)})$. By construction, it has bad reduction exactly at the prime divisors of $3d$.

For instance, consider the twist $X^{(3)}$. Since this has good reduction away from 3, the arguments of Section 3 apply to determine the associated newform. Recall the Größencharakter ψ_f associated to the newform f . Counting points mod 7, we find that $X^{(3)}$ corresponds to the twist $\psi_f \otimes \left(\frac{3}{\cdot}\right)_3$ (with trace 11 at 7). We now give the general statement:

Theorem 8.1

$X^{(d)}$ corresponds to the twist $\psi_f \otimes \left(\frac{d}{\cdot}\right)_3$, i.e. $L(T_{X^{(d)}}, s) = L(\psi_f \otimes \left(\frac{d}{\cdot}\right)_3, s)$.

The proof of this theorem is organized as follows: We exhibit another extremal elliptic fibration on the surfaces $X^{(d)}$. This will have fibres with CM by $\mathbb{Q}(\sqrt{-3})$ such that we can perform the twisting fibrewise. Then the claim will follow.

We want to give an elliptic fibration on $X^{(d)}$ with the component D_0 as a section. Therefore, we work with the affine equation of the first blow-up of the Weierstrass model (9) at the D_3 -singularity:

$$y'^2 + d^2 s^2 x' y' = s x' (x' + 1)^2. \quad (10)$$

Then $D_0 = \{s = y' = 0\}$, so the fibration is the affine projection on the x' -coordinate. We employ the usual notation, i.e. replace x' by t and likewise for s and y' . Then equation (10) becomes

$$y^2 + d^2 t x^2 y = t(t+1)^2 x.$$

In order to obtain a projective model for this, we first homogenize

$$s^3 y^2 z + d^2 s^2 t x^2 y = t(t+s)^2 x z^2.$$

Then the change of variable $y \mapsto \frac{t+s}{s} y$ gives

$$X^{(d)} : s(t+s)y^2 z + d^2 s t x^2 y = t(t+s)x z^2. \quad (11)$$

This has six singularities, two in each fibre above 0, -1 and ∞ . Their resolution produces three fibres of type IV^* . These can be read off directly from the original fibration. We sketch this in Figure 2.

The sections of the new fibration are D_0, Θ_3 and Θ_9 , as printed in blue, and the remaining components form the three singular fibres of type IV^* . In particular, they are all defined over \mathbb{Q} , as we have seen before.

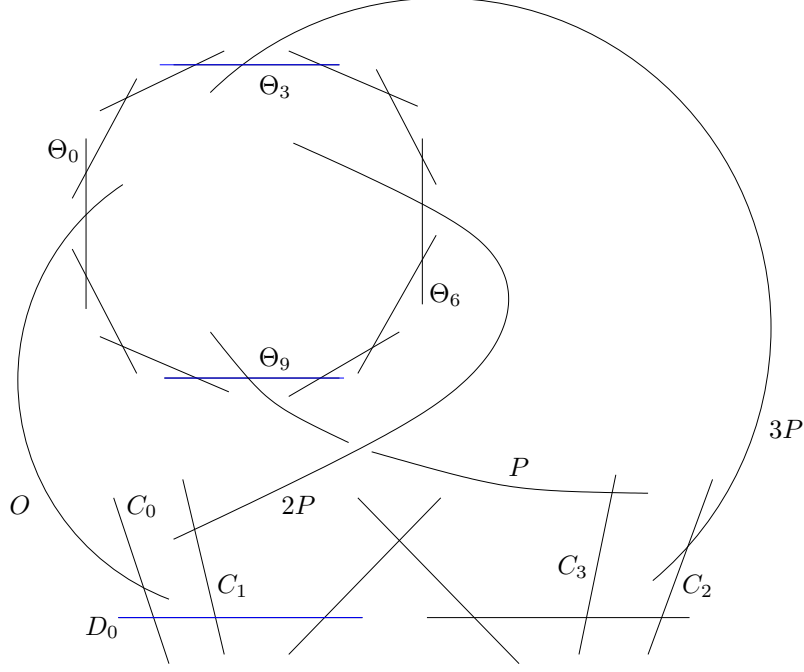


Fig. 2: The fibration on $X^{(d)}$ with three fibres of type IV^*

The smooth fibres of the above fibration are elliptic curves with CM by $\mathbb{Q}(\sqrt{-3})$. However, the impact of twisting on the associated newforms is not visible so far. Therefore, we transform eq. (11) into Weierstrass form. The procedure from [Ca, §8] gives

$$X^{(d)} : y^2 + d^2 s^2 t^2 (s+t)^2 y = x^3.$$

By [I-R, 18, Thm. 4], the Größencharaktere associated to the smooth fibres are twisted by $\left(\frac{d}{\cdot}\right)_3$ upon moving from $X^{(1)}$ to $X^{(d)}$. Using the Lefschetz fixed point formula, we can express the trace of Frobenius on $T_{X^{(d)}}$ as the sum of the traces on the smooth fibres. Hence Theorem 8.1 follows.

Remark 8.2

From the classification of [S2], it follows that any newform of weight 3 with rational coefficients and CM by $\mathbb{Q}(\sqrt{-3})$ can be realized geometrically by some twist of X .

For the only other CM-field with non-quadratic twists, $\mathbb{Q}(\sqrt{-1})$, the corresponding statement can be established using the Fermat quartic in \mathbb{P}^3 .

Corollary 8.3

Up to the bad Euler factors, we have

$$\zeta(X^{(d)}, s) \stackrel{\circ}{=} \zeta(s) \zeta(s-1)^{20} L(\psi_f \otimes \left(\frac{d}{\cdot}\right)_3, s) \zeta(s-2).$$

Acknowledgement: This paper is condensed from the last chapter of my PhD-thesis. I would like to express my gratitude to my advisor K. Hulek for all his support and encouragement during this period. I thank B. van Geemen and T. Shioda for many useful comments and discussions.

Generous support from the DFG-Schwerpunkt 1094 "Globale Methoden in der komplexen Geometrie" and the network Arithmetic Algebraic Geometry, a Marie Curie Research Training Network, is gratefully acknowledged. The paper was finished while I enjoyed the hospitality of the Dipartimento di Matematica "Frederico Enriques" of Milano University. I am very grateful for the stimulating atmosphere.

References

- [A] Artin, M.: *Supersingular K3 surfaces*, Ann. scient. Éc. Norm. Sup. (4) **7** (1974), 543-568.
- [Ca] Cassels, J. W. S.: *Lectures on Elliptic Curves*, London Math. Soc. Student Texts **24**, Cambr. Univ. Press (1991).
- [DK] Dolgachev, I., Kondō, S.: *A supersingular K3 surface in characteristic 2 and the Leech lattice*, Int. Math. Res. Not. **1** (2003), 1-23.
- [I-R] Ireland, K., Rosen, M.: *A Classical Introduction to Modern Number Theory*, Graduate Texts in Math. **84**, Springer (1982).
- [I] Ito, H.: *On extremal elliptic surfaces in characteristic 2 and 3*, Hiroshima Math. J. **32** (2002), pp. 179-188.
- [L] Livné, R.: *Motivic Orthogonal Two-dimensional Representations of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$* , Israel J. of Math. **92** (1995), 149-156.
- [LLR] Liu, Q., Lorenzini, D., Raynaud, M.: *On the Brauer group of a surface*, Inv. Math. **159** (2005), 673-676.
- [MP] Miranda, R., Persson, U.: *On Extremal Rational Elliptic Surfaces*, Math. Z. **193** (1986), 537-558.
- [S1] Schütt, M.: *Elliptic fibrations of some extremal K3 surfaces*, Rocky Mountain Journal of Mathematics **37**, 2 (2007), 609-652.
- [S2] Schütt, M.: *CM newforms with rational coefficients*, to appear in Ramanujan Journal, preprint(2005), arXiv: math.AG/0511228.
- [S3] Schütt, M.: *Hecke eigenforms and the arithmetic of singular K3 surfaces*, PhD-thesis, Univ. Hannover (2006).
- [Sh1] Shioda, T.: *On the Mordell-Weil lattices*, Comm. Math. Univ. St. Pauli, **39** (1990), 211-240.
- [Sh2] Shioda, T.: *The elliptic K3 surfaces with a maximal singular fibre*, C. R. Acad. Sci. Paris, Ser. I, **337** (2003), 461-466.
- [Si] Silverman, J. H.: *Advanced Topics in the Arithmetic of Elliptic Curves*, Graduate Texts in Math. **151**, Springer (1994).
- [S-Z] Shimada, I., Zhang, D. Q.: *Classification of extremal elliptic K3 surfaces and fundamental groups of open K3 surfaces*, Nagoya Math. J. **161** (2001), 23-54.
- [T1] Tate, J.: *On the conjectures of Birch and Swinnerton-Dyer and a geometric analog*, in: A. Grothendieck, N. H. Kuiper (eds.), *Dix exposés sur la cohomologie des schemas*, 189-214. North-Holland Publ., Amsterdam, 1968.
- [T2] Tate, J.: *Conjectures on algebraic cycles in ℓ -adic cohomology*, in: Proc. Sympos. Pure Math., Vol. **55**, 71-83. Providence, RI: Amer. Math. Soc., 1994.

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